TECHNISCHE UNIVERSITÄT DARMSTADT

UPDATED REDUCED MAGNETIC VECTOR POTENTIAL METHOD

for Superconducting Accelerator Magnets

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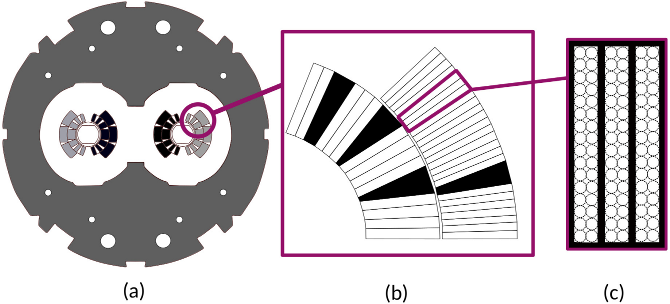
SUPERCONDUCTING ACCELERATOR MAGNET = MULTI-SCALE PROBLEM



- **Problem:** Meshing all geometrical details within a finite-element method is too expensive!
- **Idea:** Do not resolve the coils/cables/wires in the FE mesh itself, but as a separate set of threads & compute their contribution by Biot-Savart's law

OUTLINE

Reduced magnetic vector potential formulations

↓

Numerical studies

↓

Quadrupole magnet simulation

↓

Expansion to superconducting effects

↓

Summary & Outlook

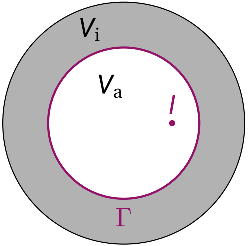
OUTLINE

- Reduced magnetic vector potential formulations
- Numerical studies
- Quadrupole magnet simulation
- Expansion to superconducting effects
- Summary & Outlook

PHYSICAL PROBLEM

We want to solve:

$$\nabla \times \left(\nu \nabla \times \vec{A}\right) = \vec{J} \qquad \text{in } V,$$
$$\vec{n} \times \vec{A} = 0 \qquad \text{on } \partial V.$$



V consists of

- V_a: air domain with source currents
- V_i: source-free domain, typically iron yoke
- Γ: interface surface between V_a and V_i

REDUCED MAGNETIC VECTOR POTENTIAL (RMVP) ANSATZ

Split the magnetic vector potential (MVP):

$$\vec{A}(\vec{r},t) = \underbrace{\vec{A}_\mathrm{s}(\vec{r})}_{\text{source MVP}} + \underbrace{\vec{A}_\mathrm{r}(\vec{r})}_{\text{reduced MVP}}$$

Compute a Biot-Savart integral for each thread $\mathcal{L}'$:

$$\vec{A}_\mathrm{s}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{L}'} \frac{I}{|\vec{r}-\vec{r}'|} \,\mathrm{d}\vec{r}'$$

Compute $\vec{A}_\mathrm{r}$ using the finite-element method:

$$\vec{A}_\mathrm{r}(\vec{r}) \approx \sum_{j=1}^{N_\mathrm{edge}} \widehat{a}_{\mathrm{s},j}\, \vec{w}_j(\vec{r})$$

STANDARD RMVP FORMULATION

CHRISTIAN PAUL 1997

1. Evaluate the source MVP $\vec{A}_\mathrm{s}$ via Biot-Savart for each point $\vec{r} \in V$.

2. Find the reduced MVP $\vec{A}_\mathrm{r} \in H_\mathrm{r}(\mathrm{curl}; V) = \left\{ \vec{A}_\mathrm{r} \in H(\mathrm{curl}; V) \ : \ \vec{n} \times \vec{A}_\mathrm{r} = -\vec{n} \times \vec{A}_\mathrm{s} \text{ on } \partial V \right\}$, s.t.

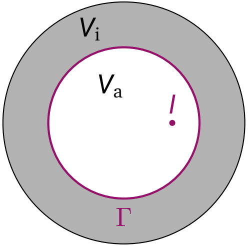
$$\left(\nu \nabla \times \vec{A}_\mathrm{r}, \nabla \times \vec{A}'_\mathrm{r}\right)_V = -\left(\nu \nabla \times \vec{A}_\mathrm{s}, \nabla \times \vec{A}'_\mathrm{r}\right)_{V_\mathrm{i}} \quad \forall \vec{A}'_\mathrm{r} \in H_\mathrm{r}(\mathrm{curl}; \mathrm{V})$$

3. Compose total MVP: $\vec{A} = \vec{A}_\mathrm{s} + \vec{A}_\mathrm{r}$ in V.

$\rightarrow$ Can get computationally expensive!

UPDATED RMVP FORMULATION: DERIVATION

D'ANGELO ET AL. 2024



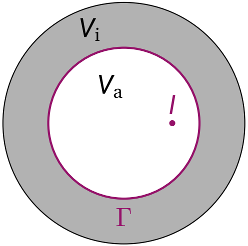
Goal: Have $\vec{A}_\mathrm{s}$ only in V_a.

Express the boundary value problem for V_a and V_i separately and introduce the source MVP $\vec{A}_\mathrm{s}$ in V_a:

$$
\begin{aligned}
\nabla \times \left(\nu_0 \nabla \times \left(\vec{A}_\mathrm{a} - \vec{A}_\mathrm{s}\right)\right) &= 0 && \text{in } V_\mathrm{a},\\
\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{i}\right) &= 0 && \text{in } V_\mathrm{i},\\
\vec{n} \times \vec{A}_\mathrm{a} &= \vec{n} \times \vec{A}_\mathrm{i} && \text{at } \Gamma,\\
\vec{n} \times \vec{H}_\mathrm{a} &= \vec{n} \times \vec{H}_\mathrm{i} && \text{at } \Gamma,\\
\vec{n} \times \vec{A}_\mathrm{i} &= 0 && \text{at } \partial V.
\end{aligned}
$$

UPDATED RMVP FORMULATION: DERIVATION

D'ANGELO ET AL. 2024



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\vec{n} \times \vec{A}_\mathrm{i} &= 0 && \text{at } \partial V.
\end{aligned}
$$

$\rightarrow$ **Problem**: The separate solutions $\vec{A}_\mathrm{a} - \vec{A}_\mathrm{s}$ in V_a and $\vec{A}_\mathrm{i}$ in V_i are not continuous at Γ!

UPDATED RMVP FORMULATION: DERIVATION

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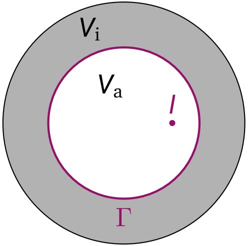

Solution: Enforce continuity by introducing the image MVP $\vec{A}_\mathrm{m}$ with $\vec{n} \times \vec{A}_\mathrm{m} = -\vec{n} \times \vec{A}_\mathrm{s}$ on Γ,

$$\begin{aligned}
\nabla \times \left(\nu_0 \nabla \times (\vec{A}_\mathrm{a} - \vec{A}_\mathrm{s} - \vec{A}_\mathrm{m})\right) &= 0 && \text{in } V_\mathrm{a}, \\
\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{i}\right) &= 0 && \text{in } V_\mathrm{i}, \\
\vec{n} \times \vec{A}_\mathrm{a} &= \vec{n} \times \vec{A}_\mathrm{i} && \text{at } \Gamma, \\
\vec{n} \times \vec{H}_\mathrm{a} &= \vec{n} \times \vec{H}_\mathrm{i} && \text{at } \Gamma, \\
\vec{n} \times \vec{A}_\mathrm{i} &= 0 && \text{at } \partial V.
\end{aligned}$$

UPDATED RMVP FORMULATION: DERIVATION

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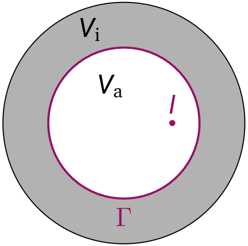

Solution: Enforce continuity by introducing the image MVP $\vec{A}_\mathrm{m}$ with $\vec{n} \times \vec{A}_\mathrm{m} = -\vec{n} \times \vec{A}_\mathrm{s}$ on Γ, and add zero:

$$
\begin{aligned}
\nabla \times \left(\nu_0 \nabla \times (\vec{A}_\mathrm{a} - \vec{A}_\mathrm{s} - \vec{A}_\mathrm{m})\right) &= 0 && \text{in } V_\mathrm{a}, \\
\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{i}\right) &= 0 && \text{in } V_\mathrm{i}, \\
\vec{n} \times \vec{A}_\mathrm{a} - \underbrace{\vec{n} \times (\vec{A}_\mathrm{s} + \vec{A}_\mathrm{m})}_{=0} &= \vec{n} \times \vec{A}_\mathrm{i} && \text{at } \Gamma, \\
\vec{n} \times \vec{H}_\mathrm{a} \underbrace{- \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m}) + \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m})}_{=0} &= \vec{n} \times \vec{H}_\mathrm{i} && \text{at } \Gamma, \\
\vec{n} \times \vec{A}_\mathrm{i} &= 0 && \text{at } \partial V.
\end{aligned}
$$

UPDATED RMVP FORMULATION: DERIVATION

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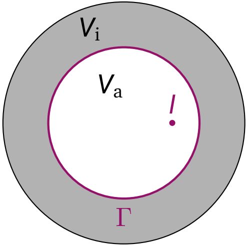

Summarize with $\vec{A}'_\mathrm{a} = \vec{A}_\mathrm{a} - \vec{A}_\mathrm{s} - \vec{A}_\mathrm{m}$:

$$\nabla \times \left(\nu_0 \nabla \times \vec{A}'_\mathrm{a}\right) = 0 \qquad \text{in } V_\mathrm{a},$$
$$\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{i}\right) = 0 \qquad \text{in } V_\mathrm{i},$$
$$\vec{n} \times \vec{A}'_\mathrm{a} = \vec{n} \times \vec{A}_\mathrm{i} \qquad \text{at } \Gamma,$$
$$\vec{n} \times \vec{H}'_\mathrm{a} - \vec{n} \times \vec{H}_\mathrm{i} = \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m}) \qquad \text{at } \Gamma,$$
$$\vec{n} \times \vec{A}_\mathrm{i} = 0 \qquad \text{at } \partial V.$$

- Sub-domain solutions $\vec{A}'_\mathrm{a}$ and $\vec{A}_\mathrm{i}$ are tangentially continuous at Γ.
- Jump of $\vec{H}$ can be interpreted as a surface current density $\vec{K}_\mathrm{g} = \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m})$.

UPDATED RMVP FORMULATION: DERIVATION

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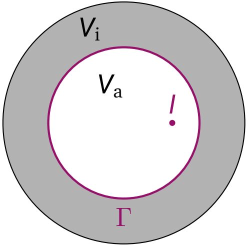

Summarize with $\vec{A}'_\mathrm{a} = \vec{A}_\mathrm{a} - \vec{A}_\mathrm{s} - \vec{A}_\mathrm{m}$:

$$\nabla \times \left(\nu_0 \nabla \times \vec{A}_\mathrm{g}\right) = 0 \qquad \text{in } V_\mathrm{a},$$
$$\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{g}\right) = 0 \qquad \text{in } V_\mathrm{i},$$
$$\vec{n} \times \vec{A}_\mathrm{g} = \vec{n} \times \vec{A}_\mathrm{g} \qquad \text{at } \Gamma,$$
$$\vec{n} \times \vec{H}_\mathrm{g} - \vec{n} \times \vec{H}_\mathrm{g} = \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m}) \qquad \text{at } \Gamma,$$
$$\vec{n} \times \vec{A}_\mathrm{g} = 0 \qquad \text{at } \partial V.$$

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UPDATED RMVP FORMULATION: DERIVATION

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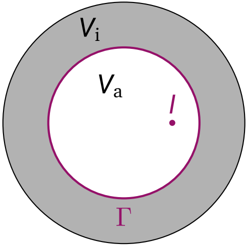

Finally, substitute

$$\vec{A}_\mathrm{g} = \begin{cases} \vec{A}'_\mathrm{a} & \text{in } V_\mathrm{a}, \\ \vec{A}_\mathrm{i} & \text{in } V_\mathrm{i}, \end{cases}$$

to obtain a single domain problem:

$$\nabla \times \left(\nu \nabla \times \vec{A}_\mathrm{g}\right) = \vec{K}_\mathrm{g} \delta_\Gamma \qquad \text{in } V,$$

$$\vec{n} \times \vec{A}_\mathrm{g} = 0 \qquad \text{at } \partial V.$$

UPDATED RMVP FORMULATION: RECIPE

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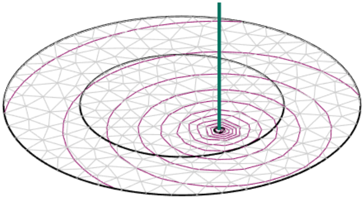
1. Evaluate the source MVP $\vec{A}_s$ via Biot-Savart only at the interface $\Gamma = \partial V_a$...
   ...and on every point of interest $\vec{r} \in V_a$.

$\rightarrow$ Huge improvement in computational efficiency!



source MVP $\vec{A}_s$ (everywhere) and source current

UPDATED RMVP FORMULATION: RECIPE

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2. Find the image MVP $\vec{A}_\mathrm{m} \in H(\mathrm{curl}; V_\mathrm{a})$, $\vec{n} \times \vec{H}_\mathrm{m} \in H^{-1/2}(\mathrm{curl}; \Gamma)$, s.t.

$$\left(\nu_0 \nabla \times \vec{A}_\mathrm{m}, \nabla \times \vec{A}'_\mathrm{m}\right)_{V_\mathrm{a}} + \left(\vec{n} \times \vec{H}_\mathrm{m}, \vec{A}'_\mathrm{m}\right)_\Gamma = 0 \qquad \forall \vec{A}'_\mathrm{m} \in H(\mathrm{curl}; V_\mathrm{a}),$$

$$\left(\vec{A}_\mathrm{m}, \vec{n} \times \vec{H}'_\mathrm{m}\right)_\Gamma + \left(\vec{A}_\mathrm{s}, \vec{n} \times \vec{H}'_\mathrm{m}\right)_\Gamma = 0 \qquad \forall \vec{n} \times \vec{H}'_\mathrm{m} \in H^{-1/2}(\mathrm{curl}; \Gamma).$$

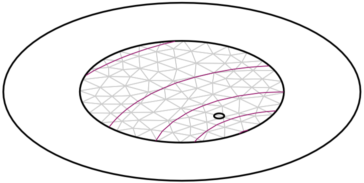

image MVP $\vec{A}_\mathrm{m}$ in V_a

UPDATED RMVP FORMULATION: RECIPE

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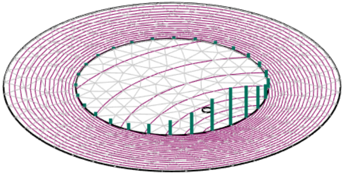
3. Find the reaction MVP $\vec{A}_\mathrm{g} \in H_0(\mathrm{curl}; V)$ s.t.

$$\left(\nu\nabla \times \vec{A}_\mathrm{g}, \nabla \times \vec{A}'_\mathrm{g}\right)_V = \left(\vec{n} \times \vec{H}_\mathrm{s}, \vec{A}'_\mathrm{g}\right)_\Gamma + \left(\vec{n} \times \vec{H}_\mathrm{m}, \vec{A}'_\mathrm{g}\right)_\Gamma \quad \forall \vec{A}'_\mathrm{g} \in H_0(\mathrm{curl}; V).$$



reaction MVP $\vec{A}_\mathrm{g}$ in V,
surface current density $\vec{K}_\mathrm{g} = \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m})$ on Γ

UPDATED RMVP FORMULATION: RECIPE

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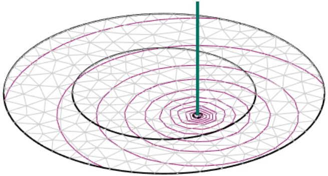

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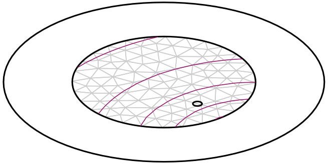

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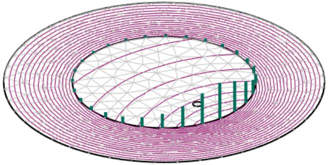
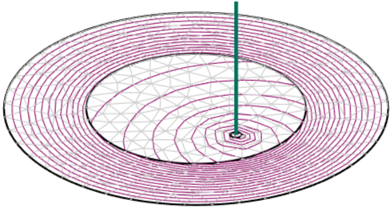


source MVP $\vec{A}_\mathrm{s}$ image MVP $\vec{A}_\mathrm{m}$ reaction MVP $\vec{A}_\mathrm{g}$

=



4. Compose final solution:

$$\vec{A} = \vec{A}_\mathrm{s} + \vec{A}_\mathrm{m} + \vec{A}_\mathrm{g} \quad \text{in } V_\mathrm{a},$$
$$\vec{A} = \vec{A}_\mathrm{g} \quad \text{in } V_\mathrm{i}.$$

OUTLINE

Reduced magnetic vector potential formulations

↓

Numerical studies

↓

Quadrupole magnet simulation

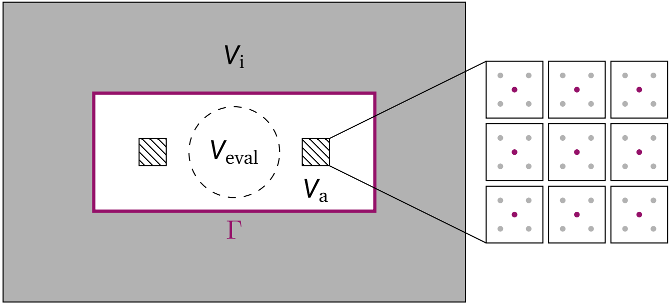
↓

Expansion to superconducting effects

↓

Summary & Outlook

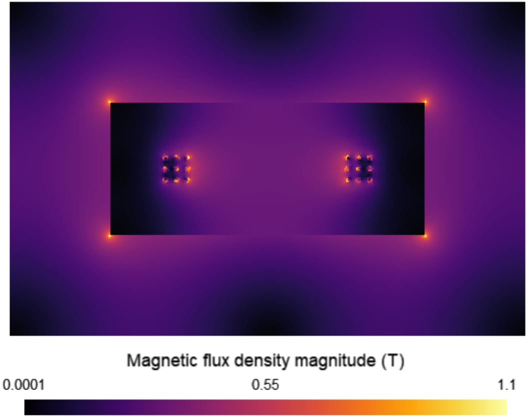
RACETRACK COIL MODEL



- Discretization of windings as set of line currents
- Simplest case: single line currents at center points

- Implementation of the RMVP formulation in the open-source FE solver GetDP
- 2D linear magnetostatic simulation

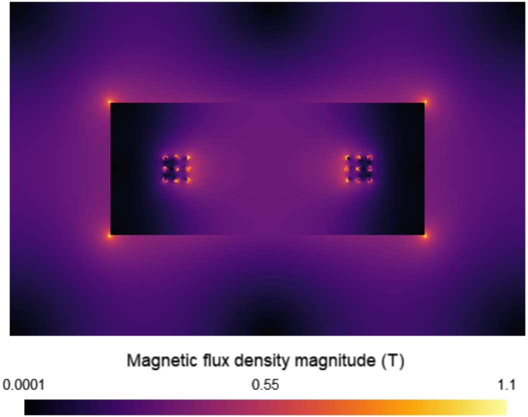
MAGNETIC FLUX DENSITY & L²-ERROR



Biot-Savart:

$$\vec{A}_s(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{L}'} \frac{I}{|\vec{r} - \vec{r}'|} \, d\vec{r}'$$

MAGNETIC FLUX DENSITY & L^2-ERROR



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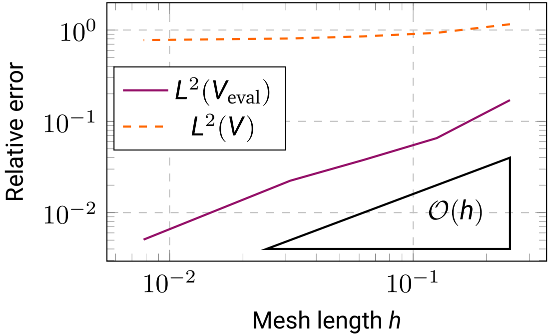
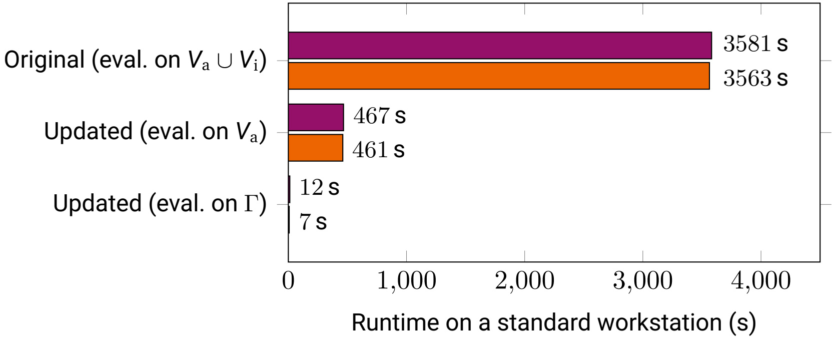


PERFORMANCE COMPARISON

BENCHMARK MODEL WITH > 100,000 DOF AND 18 WIRES



1. **Biot-Savart** computation is dominant $\Rightarrow$ Parallelize and use fast-multipole methods
2. **Updated RMVP** formulation by far superior than standard one

OUTLINE

- Reduced magnetic vector potential formulations
- Numerical studies
- **Quadrupole magnet simulation**
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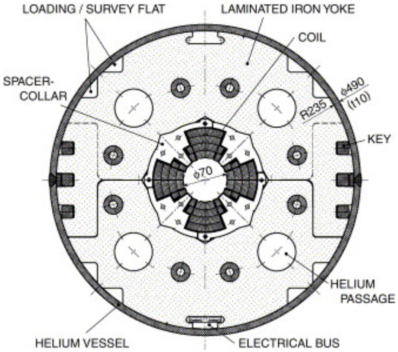
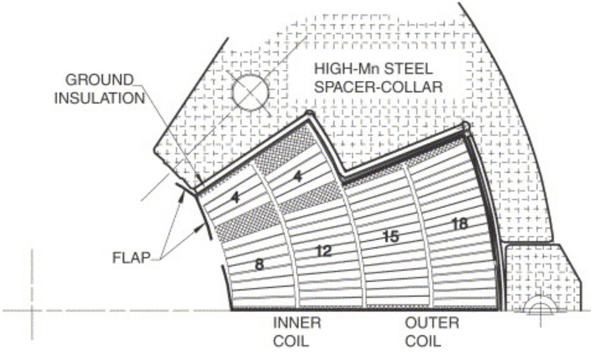
QUADRUPOLE MAGNET: SETUP

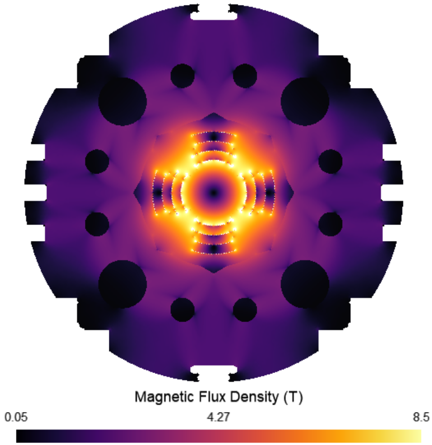


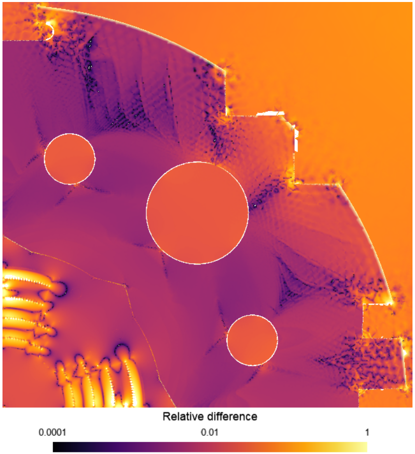


- 2D magnetostatic nonlinear simulation
- Comparison to conventional 2D FE simulation

- One line current per winding
- 488 line currents in total

QUADRUPOLE MAGNET: SIMULATION





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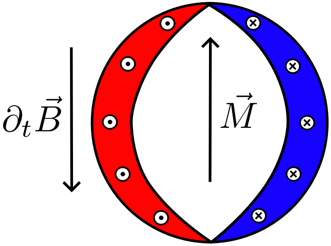
PERFECT DIAMAGNETISM OF SUPERCONDUCTORS



Time-varying external magnetic flux density $\vec{B}$

$\downarrow$ induces $\downarrow$

screening currents

$\downarrow$ generate $\downarrow$

counter-acting magnetization $\vec{M}$

- This magnetization effect has to be considered in the RMVP formulation.
- **Note**: The screening current distribution in the wire gives rise to a dipole field.

HIGHER-ORDER MAGNETIC MOMENTS

Biot-Savart:

$$\vec{A}_{\mathrm{s}}(\vec{r}) = \frac{\mu_0}{4\pi} \int\limits_{\mathcal{V}'} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}V'$$

Multipole expansion:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\vec{r}' \cdot \nabla\right)^n \frac{1}{|\vec{r}|}$$

HIGHER-ORDER MAGNETIC MOMENTS

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$$\vec{A}_\mathrm{s}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}V'$$

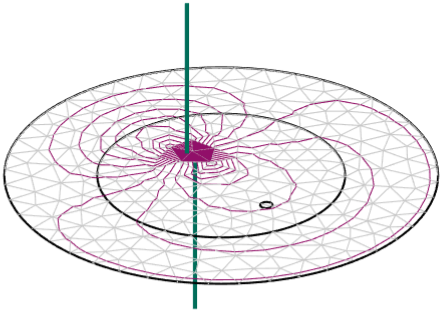
Multipole expansion:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\vec{r}' \cdot \nabla\right)^n \frac{1}{|\vec{r}|}$$

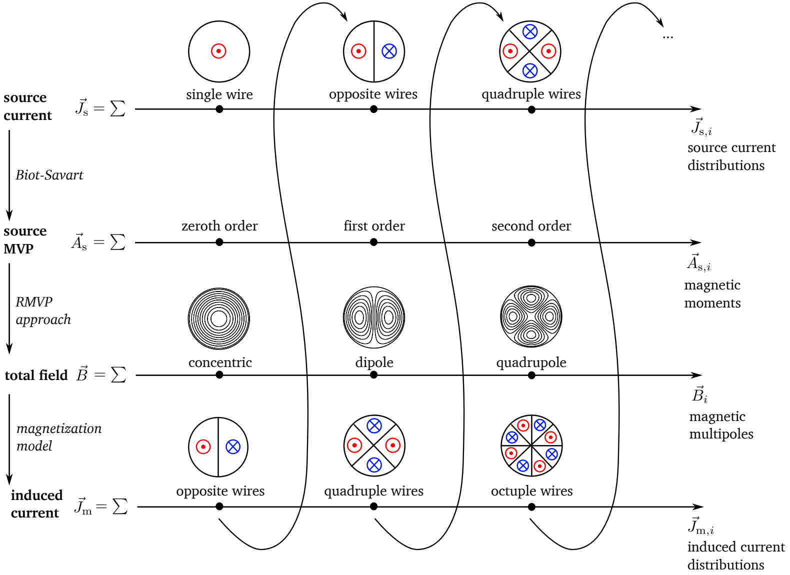
$n = 1$ (dipole):

$$\vec{A}_{\mathrm{s},1}(\vec{r}) = -\frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \vec{J}(\vec{r}') \frac{\vec{r} \cdot \vec{r}'}{|\vec{r}|^3} \, \mathrm{d}V'$$

$\hookrightarrow$ field generated by a dipole moment at $\vec{r}'$ or two opposite wires around $\vec{r}'$



HIGHER-ORDER MAGNETIC MOMENTS



SUMMARY & OUTLOOK

Summary:

- RMVP ansatz: No explicit meshing of wires in the FE mesh
- Update: Evaluate Biot-Savart's law only on V_a at most
- High efficiency gain compared to original RMVP approach
- Consider screening currents by a dipole moment

↪ QR code to paper

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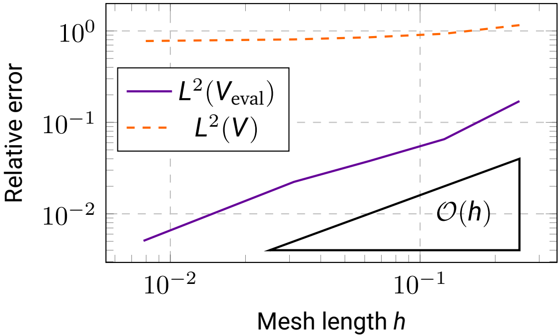
Outlook:

- Implement higher-order magnetic moments to consider magnetization and eddy current effects
- Adapt RMVP approach for high-temperature superconducting coils (tapes!)

BACKUP SLIDES

RACETRACK COIL: CONVERGENCE STUDY

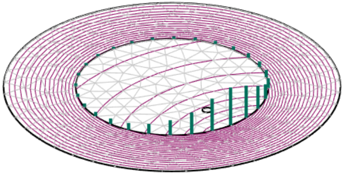
- L^2-error w.r.t. reference solution
- Choose V_{eval} s.t. $\mathcal{L}' \not\subseteq V_{\mathrm{eval}}$
- **Expectation** for lowest-order FEs: Quadratic convergence of L^2-error ($\rightarrow$ *Aubin-Nitsche lemma*)
- **Observation**: Linear convergence



DIRAC SOURCE TERM AS PROBLEM?

3. Find the reaction MVP $\vec{A}_\mathrm{g} \in H_0(\mathrm{curl}; V)$ s.t.

$$\left(\nu \nabla \times \vec{A}_\mathrm{g}, \nabla \times \vec{A}'_\mathrm{g}\right)_V = \left(\vec{n} \times \vec{H}_\mathrm{s}, \vec{A}'_\mathrm{g}\right)_\Gamma + \left(\vec{n} \times \vec{H}_\mathrm{m}, \vec{A}'_\mathrm{g}\right)_\Gamma \quad \forall \vec{A}'_\mathrm{g} \in H_0(\mathrm{curl}; V).$$



reaction MVP $\vec{A}_\mathrm{g}$ in V,
surface current density
$\vec{K}_\mathrm{g} = \vec{n} \times (\vec{H}_\mathrm{s} + \vec{H}_\mathrm{m})$ on Γ

- **Hypothesis**: $\vec{J}_\mathrm{g} = \vec{K}_\mathrm{g}\delta_\Gamma$ as Dirac source term makes problem not regular
- **Literature**: Linear convergence of the L^2-error of 2D elliptic problems with Dirac source term (Scott 1973)
  - Even $\mathcal{O}(h^{1/2})$ in 3D $\leftarrow$ to check!